Geodynamics
Lecture 9
Basics of fluid mechanics

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Paper discussion on Thursday

Himalayan tectonics explained by extrusion of a low-viscosity crustal channel coupled to focused surface denudation

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- Paper discussion on Thursday after lecture
- Similar to last discussion
  - Read paper in advance
  - Be prepared to present a figure to the class (in groups)
Goals of this lecture

- Introduce the basic concepts of fluid mechanics
- Investigate examples of fluid flow in one dimension
The fluids and the Earth

- **Fluid**: Any material that flows in response to an applied stress

- Differences between **solids** and **fluids**

<table>
<thead>
<tr>
<th>Solids</th>
<th>Fluids</th>
</tr>
</thead>
<tbody>
<tr>
<td>Strain from being stressed</td>
<td>Continuous deformation under applied forces</td>
</tr>
<tr>
<td>Stresses related to strains</td>
<td>Stresses related to rates of strain</td>
</tr>
<tr>
<td>Strain result of displacement gradients</td>
<td>Strain result of velocity gradients</td>
</tr>
</tbody>
</table>

- **Rheological** (or **constitutive**) **law**: An equation relating stress to strain rates in a fluid
A **Newtonian fluid** is a fluid in which there is a linear relationship between the rate of strain and the applied stress.

**What would this relationship look like as an equation?**
Newtonian (or linear) fluid

- A **Newtonian fluid** is a fluid in which there is a linear relationship between the rate of strain and the applied stress.

  - What would this relationship look like as an equation?

  \[ \sigma \propto \dot{\varepsilon} \quad \text{or} \quad \sigma = \eta \dot{\varepsilon} \]

- The proportionality constant \( \eta \) is known as the **dynamic (or shear) viscosity**.

- Dynamic viscosity has units of \( \text{Pa s} \).
Fluid mechanics

- The concepts of fluid mechanics are based on conservation of
  - mass
  - momentum, and
  - energy
- Conservation of mass, momentum and energy are combined with rheological laws to describe fluid movement under an applied force
Hydrothermal fluid flow

Geysir in Yellowstone National Park, USA
Magma migration and flow
Mantle convection

Fig. 1.61, Turcotte and Schubert, 2014
The most simple fluid flow we can consider is flow of a fluid in one direction within a channel of fixed width. This kind of flow might occur, for example, as a result of plates moving over the asthenosphere.
1D channel flows

- Fluid is flowing with velocity $u$ in the $x$ direction, and the flow velocity $u$ is a function of distance across the channel $y$.
- Flow results from:
  - a pressure gradient $(p_0 - p_1)/l$, and/or
  - motion of the side wall of the channel $u_0$.

Fig. 6.1, Turcotte and Schubert, 2014
ID channel flows

- **Shear**, or a gradient in the velocity, in the channel results in a shear stress $\tau$ that is exerted on horizontal planes in the fluid.

- For a Newtonian fluid with a constant dynamic viscosity $\eta$ we can state

  $$\tau = \eta \frac{du}{dy}$$
1D channel flows

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Fig. 6.1, Turcotte and Schubert, 2014
ID channel flows

We can now determine the flow in the channel using the equation of motion, based on the force balance on a layer of fluid of thickness $\delta y$ and length $l$.

The net pressure force on the element in the $x$ direction is

$$(p_1 - p_0)\delta y$$
Because the shear stress $\tau$ and velocity $u$ are both only a function of distance $y$, the **shear force** on the upper boundary of the element is

$$-\tau(y)l$$

The equivalent **shear force** on the lower boundary is

$$\tau(y + \delta y)l = \left(\tau(y) + \frac{d\tau}{dy}\delta y\right)l$$

**Fig. 6.1, Turcotte and Schubert, 2014**
The net force (or sum of the forces) must be equal to zero, or

\[(p_1 - p_0)\delta y + \left[ \tau(y) + \frac{d\tau}{dy}\delta y \right] l - \tau(y)l = 0\]

As \(\delta y \rightarrow 0\), the relationship above becomes

\[\frac{d\tau}{dy} = -\frac{(p_1 - p_0)}{l}\]
The right side of the previous equation is the **horizontal pressure gradient** in the channel:

\[
\frac{dp}{dx} = -\frac{(p_1 - p_0)}{\ell}
\]

From which the **equation of motion** can be written:

\[
\frac{d\tau}{dy} = \frac{dp}{dx}
\]
Velocity in the channel is found by substituting the rheological law for a Newtonian fluid into the equation of motion:

\[ \frac{d\tau}{dy} = \frac{d}{dy} \eta \frac{du}{dy} = \eta \frac{d^2 u}{dy^2} = \frac{dp}{dx} \]

Integrating the equation above twice yields:

\[ u = \frac{1}{2\eta} \left( \frac{dp}{dx} \right) y^2 + c_1 y + c_2 \]
The constants $c_1$ and $c_2$ can be found by applying the boundary conditions that $u = 0$ at $y = h$, and $u = u_0$ at $y = 0$ (no slip).
OK, now what does this equation tell us?

\[
u = \frac{1}{2\eta} \frac{dp}{dx} (y^2 - hy) - \frac{u_0 y}{h} + u_0
\]

- We’ll now look at two simple end-member fluid flow behaviors
  - **Couette flow**: Zero pressure gradient \((dp/dx = 0)\)
  - **Poiseuille flow**: Zero boundary velocity \((u_0 = 0)\)

- **What are your predictions for what these flows should look like?**
A **Couette flow** has no pressure gradient, or $dp/dx = 0$, reducing the 1D equation for velocity in the channel down to

$$u = u_0 \left(1 - \frac{y}{h}\right)$$

Clearly, this predicts a linear increase in velocity from $y = h$ to $y = 0$. 
Poiseuille flow

- **Poiseuille flow** is driven only by a pressure gradient in the channel with zero boundary velocities \((u_0 = 0)\), thus

\[
u = \frac{1}{2\eta} \frac{dp}{dx} \left( y^2 - hy \right)
\]

- In a coordinate system with \(y'\) at the middle of the channel we can say \(y' = y - h/2\), which results in the relationship

\[
u = \frac{1}{2\eta} \frac{dp}{dx} \left( y'^2 - \frac{h^2}{4} \right)
\]
**Application: Salt tectonics**

- Rock salt is a common rock type in sedimentary basins that is nearly incompressible and can be modeled as a Newtonian fluid with a low viscosity ($\eta \approx 10^{18}$ Pa s; it flows readily).
- This flow and tendency of salt to migrate when loaded by sedimentary overburden is the focus of studies of salt tectonics.
- Let’s now consider a simple set of experiments for a 2D continental passive margin overlying a layer of rock salt.
Application: Salt tectonics

- In this scenario, we might expect different flow behavior in the rock salt depending on the deformation of the overlying sediment.

- We predict **Poiseuille flow** when the sedimentary overburden is stable and does not move horizontally.
Application: Salt tectonics

- When the sedimentary overburden cannot support the lateral stress due to variations in its thickness, failure will occur in the sediments, leading to horizontal translation of the sediment.
- This produces a dominantly **Couette-type** of flow in the salt.
Application: Salt tectonics

In nature, it is likely that salt flows in this type of environment involve both **Couette** and **Poiseuille** components, resulting in a velocity field that is a combination of both.
Application: Salt tectonics

Numerical model predictions for variable sediment strength

Figure 5c shows a model with a thin (1.5 km) downdip overburden. In this case, the Couette velocity caused by the unstable, moving overburden is significantly greater than the Poiseuille velocity and a linear velocity profile develops in the viscous layer. Thus the numerical model results conform to the conceptual flow regimes illustrated in Fig. 1.

COMPARISON OF ANALYTICAL AND NUMERICAL RESULTS

Stability results

The stability criterion defined by Eqn. (8) is shown as a solid curve (Fig. 6) as a function of the internal angle of friction \( \theta \) and the downdip overburden thickness, \( h_2/C_3 \), for a constant value of the updip overburden thickness, \( h_1/C_3 = 4.5 \). For a given internal angle of friction, the minimum value of \( h_2/C_3 \) is needed for the overburden to remain stable. For progressively higher internal angles of friction, the overburden strength increases and the \( h_2/C_3 \) needed to keep the overburden stable decreases.

To test the response of the finite-element model against the stability criterion, we examine sets of models in which \( h_1/C_3 \) is held constant, and \( h_2/C_3 \) is varied for a given overburden strength \( f \). Numerical model sets of this type span \( f = 4 \) to \( f = 50 \) (Fig. 6) to yield a suite of model results for comparison with the non-dimensional analytical stability criterion. The numerical model results are converted to the non-dimensional form using \( h_c \) and \( k \) as defined in Eqn. (8). The results are coded according to the initial velocity pattern predicted for the first 10 time steps (before significant changes in the geometry occur) (Fig. 6). The overall results show a good agreement between the finite-element models and the analytically predicted stability criterion. This indicates that the numerical model is capable of calculating stresses and overburden stability associated with flows caused by differential loading. It is not possible to determine the absolute accuracy of the numerical results from the comparison of the numerical model and the analytical predictions because the analytical theory is itself approximate.

Initial velocities

A suite of similar numerical models was compared with the analytical Couette velocity prediction of Eqn. (12). For these models, \( h_1/C_3 = 4.5 \) and the internal angle of friction \( \theta \) is varied. The results are shown in Fig. 6, where the initial velocity pattern is predicted for the first 10 time steps (before significant changes in the geometry occur) (Fig. 6). The overall results show a good agreement between the finite-element models and the analytically predicted stability criterion. This indicates that the numerical model is capable of calculating stresses and overburden stability associated with flows caused by differential loading. It is not possible to determine the absolute accuracy of the numerical results from the comparison of the numerical model and the analytical predictions because the analytical theory is itself approximate.
• One model for mantle flow is that the motion of lithospheric plates on the Earth’s surface produces a **counterflow** in the uppermost asthenosphere (upper ~100-200 km).

• If we assume the plate is rigid and moving at velocity $u_0$, and that the velocity at some depth $y = h$ must be zero, it is clear that the counterflow is **opposite** in direction to the plate motion in order to **conserve mass**.
Asthenospheric counterflow

- Mathematically, we can state that as

\[ u_0 h_L + \int_0^h u \, dy = 0 \]

where \( h_L \) is the thickness of the lithosphere and \( h \) is the thickness of the asthenosphere.

- If we insert our equation for 1D channel flow in the second term, we get

\[ u_0 h_L + \frac{h^3}{12\eta} \frac{dp}{dx} + \frac{u_0 h}{2} = 0 \]
Asthenospheric counterflow

- If we now solve for the pressure gradient, we find
  \[
  \frac{dp}{dx} = \frac{12\eta u_0}{h^2} \left( \frac{h_L}{h} + \frac{1}{2} \right)
  \]

- And this can be inserted into the equation for 1D channel flow to get the predicted velocity profile for a counterflow
  \[
  u = u_0 \left[ 1 - \frac{y}{h} + 6 \left( \frac{h_L}{h} + \frac{1}{2} \right) \left( \frac{y^2}{h^2} - \frac{y}{h} \right) \right]
  \]
• Looking at this equation for a moment, what strikes you as perhaps somewhat surprising?

• Is there anything missing that you might expect to see?
We’ve dealt thus far with the dynamic viscosity $\eta$, but a similar quantity, the **kinematic viscosity** $\nu$, can be quite useful. One reason this is useful is the units. $\nu$ has units of $m^2 s^{-1}$, meaning it can be thought of as a **diffusivity for momentum**, much like $\kappa$, the thermal diffusivity.

\[
\nu = \frac{\eta}{\rho}
\]
Kinematic viscosity and the Prandtl number

The Prandtl number is the ratio of the kinematic viscosity to the thermal diffusivity, giving us a relationship between thermal diffusion and diffusion of momentum.

\[ Pr \equiv \frac{\nu}{\kappa} \]

A fluid with a large Prandtl number will diffuse momentum more quickly than heat and the opposite is true for a fluid with a small Prandtl number.

<table>
<thead>
<tr>
<th>Fluid</th>
<th>Kinematic Viscosity ( \nu ) (m² s⁻¹)</th>
<th>Thermal Diffusivity ( \kappa ) (m² s⁻¹)</th>
<th>Prandtl Number ( Pr )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Air</td>
<td>( 1.45 \times 10^{-5} )</td>
<td>( 2.02 \times 10^{-5} )</td>
<td>0.72</td>
</tr>
<tr>
<td>Water</td>
<td>( 1.14 \times 10^{-6} )</td>
<td>( 1.40 \times 10^{-7} )</td>
<td>8.1</td>
</tr>
<tr>
<td>Mercury</td>
<td>( 1.16 \times 10^{-7} )</td>
<td>( 4.2 \times 10^{-6} )</td>
<td>0.028</td>
</tr>
<tr>
<td>Ethyl alcohol</td>
<td>( 1.70 \times 10^{-6} )</td>
<td>( 9.9 \times 10^{-8} )</td>
<td>17.2</td>
</tr>
<tr>
<td>Carbon tetrachloride</td>
<td>( 6.5 \times 10^{-7} )</td>
<td>( 8.4 \times 10^{-8} )</td>
<td>7.7</td>
</tr>
<tr>
<td>Olive oil</td>
<td>( 1.08 \times 10^{-4} )</td>
<td>( 9.2 \times 10^{-8} )</td>
<td>1,170</td>
</tr>
<tr>
<td>Glycerine</td>
<td>( 1.85 \times 10^{-3} )</td>
<td>( 9.8 \times 10^{-8} )</td>
<td>18,880</td>
</tr>
</tbody>
</table>
Pressure drops in channels are often related to a **hydraulic head**

\[ H \equiv \frac{(p_1 - p_0)}{\rho g} \]

The **hydraulic head** is the thickness (or height) of a fluid required to generate a hydrostatic pressure equal to \( p_1 - p_0 \), the pressure drop along the channel.
Recap

- **Fluid mechanics** describes fluid motion based on the conservation of **mass**, **momentum** and **energy**

- 1D channel flows can be divided into two fundamental types that relate to the conditions in the channel

  - **Couette flow**: Linear velocity gradient across channel, pressure gradient very small

  - **Poiseuille flow**: Parabolic velocity profile across channel, minimal displacement of channel walls